Back paper Exam
Algebra-II
MMath-I, 25-26

Time: 3 hours

Total marks: 100

Answer all questions.

No marks will be awarded in absence of complete justification. Notations are as used in class.

1. (a) State and prove *orbit stabiliser theorem* for group actions. (10)

   (b) Show that a group of order 12 either contains a normal subgroup of order 3, or is isomorphic to A_4. (10)

2. (a) Prove that the alternating group A_n is simple for all $n \neq 4$. (12)

   (b) Show that A_n is the only nontrivial proper normal subgroup of S_n, for $n \neq 4$. (8)

3. (a) Show that if $\Phi_n(x)$ denotes the n-th cyclotomic polynomial, then

$$x^n - 1 = \Pi_{d/n}\Phi_d(x).$$

   (b) Show that for every prime p, and every $n \in \mathbb{N}$, there exists a unique field, say $\mathbb{F}_{p^n}$, of order p^n, upto isomorphism. Show that $\mathbb{F}_{p^n}|\mathbb{F}_p$ is a simple Galois extension.

   (c) If $n > 2$ and ζ is a primitive n-th root of unity over $\mathbb{Q}$, then show that $[\mathbb{Q}(\zeta + \zeta^{-1}) : \mathbb{Q}] = \phi(n)/2$, where ϕ denotes the Euler-ϕ function. (6+8+6)

4. (a) Show that if K/F is a finite extension, then K has only finitely many intermediate fields over F if and only if K is a simple extension over F.

   (b) State and prove *Primitive Element Theorem* for finite separable extensions.

   (c) Give an example to show that finite inseparable extensions need not be simple. (8+4+8)

5. Fix $n \in \mathbb{N}$. Let F be a field such that char(F) does not divide n, and F contains all n-th roots of unity.

   (a) Show that the simple extension $F(\sqrt[n]{a})$ for some $a \in F$ is a a cyclic extension over F, of degree divining n.

   (b) Let K be any cyclic extension of degree n over F. Then show that $K = F(\sqrt[n]{a})$ for some $a \in F$.

   (c) Find the Galois group of the polynomial $f(x) = x^5 - 5x + 5$ over $\mathbb{Q}$. (6+8+6)

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